SUBWORD COMPLEXITY AND LAURENT SERIES WITH COEFFICIENTS IN A FINITE FIELD

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Abstract. Decimal expansions of classical constants such as $\sqrt{2}$, π and $\zeta(3)$ have long been a source of difficult questions. In the case of Laurent series with coefficients in a finite field, where no carry-over difficulties appear, the situation seems to be simplified and drastically different. On the other hand, Carlitz introduced analogs of real numbers such as π, e or $\zeta(3)$. Hence, it became reasonable to enquire how "complex" the Laurent representation of these "numbers" is.

In this paper we prove that the inverse of Carlitz's analog of π, Π_q, has in general a linear complexity, except in the case $q = 2$, when the complexity is quadratic. In particular, this implies the transcendence of Π_2 over $\mathbb{F}_2(T)$. In the second part, we consider the classes of Laurent series of at most polynomial complexity and of zero entropy. We show that these satisfy some nice closure properties.

1. Introduction and motivations

A long standing open question concerns the digits of the real number $\pi = 3.14159\cdots$. The decimal expansion of π has been calculated to billions of digits and unfortunately, there are no evident patterns occurring. Actually, for any $b \geq 2$, the b-ary expansion of π looks like a random sequence (see for instance [10]). More concretely, it is widely believed that π is normal, meaning that all blocks of digits of equal length occur in the b-ary representation of π with the same frequency, but current knowledge on this point is scarce.

A usual way to describe the disorder of an infinite sequence $\mathbf{a} = a_0a_1a_2\cdots$ is to compute its subword complexity, which is the function that associates to each positive integer m the number $p(\mathbf{a}, m)$ of distinct blocks of length m occurring in the word $\mathbf{a}$. Let α be a real number and let $\mathbf{a}$ be the representation of α in an integral base $b \geq 2$. The complexity function of α is defined as follows:

$$p(\alpha, b, m) = p(\mathbf{a}, m),$$

for any positive integer m.

Notice that π being normal would imply that its complexity must be maximal, that is $p(\pi, b, m) = b^m$. In this direction, similar questions have been asked about other well-known constants like e, $\log 2$, $\zeta(3)$ or $\sqrt{2}$ and it is widely believed that the following conjecture is true.

Conjecture 1.1. *Let α be one of the classical constants: π, e, $\log 2$, $\zeta(3)$ and $\sqrt{2}$. The complexity of the real number α satisfies:*

$$p(\alpha, b, m) = b^m,$$

for every positive integer m and every base $b \geq 2$.

We mention that in all this paper we will use Landau's notations. We write $f(m) = \Theta(g(m))$ if there exist positive real numbers k_1, k_2, n_0 such that, for every $n > n_0$ we have

$$k_1 |g(n)| < |f(n)| < k_2 |g(n)| .$$

We write also $f(m) = O(g(m))$ if there exist two positive real numbers k, n_0 such that, for every $n \geq n_0$ we have:

$$|f(n)| < k |g(n)| .$$

If α is a rational real number then $p(\alpha, b, m) = O(1)$, for every integer $b \geq 2$. Moreover, there is a classical theorem of Morse and Hedlund [28] which states that an infinite sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is eventually periodic if and only if $p(\mathbf{a}, m)$ is bounded. If not, the complexity function is strictly increasing. In particular,

$$p(\mathbf{a}, m) \geq m + 1, \tag{1}$$

for every nonnegative integer m.

A sequence which saturates the inequality above is called a Sturmian sequence (see the original papers of Morse and Hedlund [28, 29]).

According to this theorem, an irrational real number α has a complexity function which satisfies $p(\alpha, b, m) \geq m + 1$, for every $m \in \mathbb{N}$. Concerning irrational algebraic numbers, the main result known to date in this direction is due to Adamczewski and Bugeaud [3]. These authors proved that the complexity of an irrational algebraic real number α satisfies

$$\lim_{m\to\infty} \frac{p(\alpha, b, m)}{m} = +\infty,$$

for any base $b \geq 2$.

For more details about complexity of algebraic real numbers, see [3, 4]. For classical transcendental constants, there is a more ambiguous situation and, to the best of our knowledge, the only result that improves the bound following from Inequality (1) was recently proved in [1]. It concerns the real number e and some other exponential periods. More precisely, Adamczewski showed that if ξ is an irrational real number whose irrational exponent $\mu(\xi) = 2$, then

$$\lim_{m\to\infty} p(\xi, b, m) - m = +\infty,$$

for any base $b \geq 2$.

The present paper is motivated by this type of questions, but asked for Laurent series with coefficients in a finite field. In the sequel we will denote

respectively by $\mathbb{F}_q(T)$, $\mathbb{F}_q[[T^{-1}]]$ and $\mathbb{F}_q((T^{-1}))$ the field of rational functions, the ring of formal series and the field of Laurent series over the finite field $\mathbb{F}_q$, q being a power of a prime number p.

Let us also recall the well-known analogy between integers, rationals and real numbers on one side, and polynomials, rationals functions, and Laurent series with coefficients in a finite field, on the other side. Notice that, the coefficients in $\mathbb{F}_q$ play the role of "digits" in the basis given by the powers of the indeterminate T. There is still a main difference: in the case of real numbers, it is hard to control carry-overs when we add or multiply whereas in the case of power series over a finite field, this difficulty disappear.

By analogy with the real numbers, the complexity of a Laurent series is defined as the subword complexity of its sequence of coefficients. Again, the theorem of Morse and Hedlund gives a complete description of the rational Laurent series; more precisely, they are the Laurent series of bounded complexity. Hence, most interesting questions concern irrational series.

There is a remarkable theorem of Christol [18] which describes precisely the algebraic Laurent series over $\mathbb{F}_q(T)$ as follows. Let $f(T) = \sum_{n\geq -n_0} a_n T^{-n}$ be a Laurent series with coefficients in $\mathbb{F}_q$. Then f is algebraic over $\mathbb{F}_q(T)$ if, and only if, the sequence of coefficients $(a_n)_{n\geq 0}$ is p-automatic.

For more references on automatic sequences, see for example [8]. Furthermore, Cobham proved that the subword complexity of an automatic sequence is at most linear [20]. Hence, an easy consequence of those two results is the following.

Theorem 1.1. *Let $f \in \mathbb{F}_q((T^{-1}))$ algebraic over $\mathbb{F}_q(T)$. Then we have:*

$$p(f, m) = O(m).$$

The reciprocal is obviously not true, since there are uncountable many Laurent series with linear complexity. In contrast with real numbers, the situation is thus clarified in the case of algebraic Laurent series. Also, notice that Conjecture 1.1 and Theorem 1.1 point out the fact that the situations in $\mathbb{F}_q((T^{-1}))$ and in $\mathbb{R}$ appear to be completely opposite.

On the other hand, Carlitz introduced [21] functions in positive characteristic by analogy with the number π, the Riemann ζ function, the usual exponential and the logarithm function. Many of these were shown to be transcendental over $\mathbb{F}_q(T)$ (see [22, 26, 33, 34, 35]). In the present paper we focus on the analog of π, denoted, for each q, by Π_q, and we prove that its inverse has a "low" complexity. More precisely, we will prove in Section 3 the following results.

Theorem 1.2. *Let $q = 2$. The complexity of the inverse of Π_q satisfies:*

$$p\left(\frac{1}{\Pi_2}, m\right) = \Theta(m^2).$$

Theorem 1.3. *Let $q \geq 3$. The complexity of the inverse of Π_q satisfies:*

$$p\left(\frac{1}{\Pi_q}, m\right) = \Theta(m).$$

Since any algebraic series has a linear complexity (by Theorem 1.1), the following corollary yields.

Corollary 1.1. *Π_2 is transcendental over $\mathbb{F}_2(T)$.*

The transcendence of Π_q over $\mathbb{F}_q(T)$ was first proved by Wade in 1941 (see [34]) using an analog of a classical method of transcendence in zero characteristic. Another proof was given by Yu in 1991 (see [35]), using the theory of Drinfeld modules. Then, de Mathan and Cherif, in 1993 (see [22]), using tools from Diophantine approximation, proved a more general result, but in particular their result implied the transcendence of Π_q.

Christol's theorem has also been used as a combinatorial criterion in order to prove the transcendence of Π_q. This is what is usually called an "automatic proof". The non-automaticity and also the transcendence, was first obtained by Allouche, in [6], via the so-called q-kernel. Notice that our proof of transcendence here is based also by Christol's theorem, but we obtain the non-automaticity of Π_2 over $\mathbb{F}_2(T)$ as a consequence of the subword complexity.

Furthermore, motivated by Theorems 1.2, 1.3 and by Conjecture 1.1, we consider the classes of Laurent series of at most polynomial complexity $\mathcal{P}$ and of zero entropy $\mathcal{Z}$ (see Section 4), which seem to be good candidates to enjoy some nice closure properties. In particular, we prove the following theorem.

Theorem 1.4. *$\mathcal{P}$ and $\mathcal{Z}$ are vector spaces over $\mathbb{F}_q(T)$.*

Another motivation of this work is the article [11] of Beals and Thakur. These authors proposed a classification of Laurent series in function of their space or time complexity. This complexity is in fact a characteristic of the (Turing) machine that computes the coefficient a_i, if $f(T) := \sum_i a_i T^{-i}$. They showed that some classes of Laurent series have good algebraic properties (for instance, the class of Laurent series corresponding to any deterministic space class at least linear form a field). They also place some Carlitz's analogs in the computational hierarchy.

This paper is organized as follows. Some definitions and basic notions on combinatorics on words and Laurent series are recalled in Section 2. Section 3 is devoted to the study of the Carlitz's analog of π; we prove Theorems 1.2 and 1.3. In Section 4 we study some closure properties of Laurent series of "low" complexity (addition, Hadamard product, derivative, Cartier operator) and we prove Theorem 1.4; in particular, this provides a criterion of linear independence over $\mathbb{F}_q(T)$ for two Laurent series in function of their

complexity. Finally, we conclude in Section 5 with some remarks concerning the complexity of the Cauchy product of two Laurent series, which seems to be a more difficult problem.

2. Terminologies and basic notions

In this section, we briefly recall some definitions and well-known results from combinatorics on words. Moreover, we recall some basic notions on algebraic Laurent series.

A *word* is a finite, as well as infinite, sequence of symbols (or letters) belonging to a nonempty set $\mathcal{A}$, called *alphabet*. We usually denote words by juxtaposition of theirs symbols.

Given an alphabet $\mathcal{A}$, we denote by $\mathcal{A}^* := \cup_{k=0}^{\infty}\mathcal{A}^k$ the set of finite words over $\mathcal{A}$. Let $V := a_0a_1\cdots a_{m-1} \in \mathcal{A}^*$. Then the integer m is the length of V and is denoted by $|V|$. The word of length 0 is the empty word, usually denoted by ε. We also denote by $\mathcal{A}^m$ the set of all finite words of length m and by $\mathcal{A}^{\mathbb{N}}$ the set of all infinite words over $\mathcal{A}$. We typically use the uppercase italic letters X, Y, Z, U, V, W to represent elements of $\mathcal{A}^*$. We also use bold lowercase letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ to represent infinite words. The elements of $\mathcal{A}$ are usually denoted by lowercase letters $a, b, c, \cdots$.

We say that V is a *factor* (or *subword*) of a finite word U if there exist some finite words A, B, possibly empty such that $U = AVB$ and we denote it by $V \lhd U$. Otherwise, $V \ntriangleleft U$. We say that X is a *prefix* of U, and we denote by $X \prec_p U$ if there exists Y such that $U = XY$. We say that Y is a *suffix* of U, and we denote by $Y \prec_s U$ if there exists X such that $U = XY$.

Also, we say that a finite word V is a factor (or subword) of an infinite word $\mathbf{a} = (a_n)_{n\geq 0}$ if there exists a nonnegative integer j such that $V = a_ja_{j+1}\cdots a_{j+m-1}$. The integer j is called an occurence of V.

Let U, V, W be three finite words over $\mathcal{A}$, V possibly empty. We denote:

$$i(U, V, W) := \{AVB,\ A \prec_s U,\ B \prec_p W,\ A, B \text{ possibly empty}\},$$

and

$$i(U, V, W)^+ := \{AVB,\ A \prec_s U,\ B \prec_p W,\ A, B \text{ nonempty}\}.$$

If n is a nonnegative integer, we denote by $U^n := \underbrace{UU\cdots U}_{n \text{ times}}$. We denote also $U^\infty := UU\cdots$, that is U concatenated (with itself) infinitely many times. An infinite word $\mathbf{a}$ is *periodic* if there exists a finite word V such that $\mathbf{a} = V^\infty$. An infinite word is *eventually periodic* if there exist two finite words U and V such that $\mathbf{a} = UV^\infty$.

The fundamental operation on words is concatenation. Notice that $\mathcal{A}^*$, together with concatenation, form the *free monoid* over $\mathcal{A}$, whose neutral element is the empty word ε.

2.1. **Subword complexity.** Let $\mathbf{a}$ be an infinite word over $\mathcal{A}$. As already mentioned in Introduction, the *subword complexity* of $\mathbf{a}$ is the function that associates to each $m \in \mathbb{N}$ the number $p(\mathbf{a}, m)$ defined as follows:

$$p(\mathbf{a}, m) = \mathrm{Card}\{(a_j, a_{j+1}, \ldots, a_{j+m-1}),\ j \in \mathbb{N}\}.$$

For any word $\mathbf{a}$, $p(\mathbf{a}, 0) = 1$ since, by convention, the unique word of length 0 is the empty word ε.

For example, let us consider the infinite word $\mathbf{a} = aaa\cdots$, the concatenation of a letter a infinitely many times. It is obvious that $p(\mathbf{a}, m) = 1$ for any $m \in \mathbb{N}$. More generally, if $\mathbf{a}$ is eventually periodic, then its complexity function is bounded.

On the other side, let us consider the infinite word of Champernowne over the alphabet $\{0, 1, 2, 3, \ldots, 9\}$, $\mathbf{a} := 0123456789101112\cdots$. Notice that $p(\mathbf{a}, m) = 10^m$ for every positive integer m.

More generally, one can easily prove that for every $m \in \mathbb{N}$ and for every word $\mathbf{a}$ over the alphabet $\mathcal{A}$, we have the following:

$$1 \leq p(\mathbf{a}, m) \leq (\mathrm{card}\,\mathcal{A})^m.$$

We give now an important tool we shall use in general, in order to obtain a bound of the subword complexity function (for a proof see for example [8]):

Lemma 2.1. *Let $\boldsymbol{a}$ be an infinite word over an alphabet $\mathcal{A}$. We have the following properties:*

- $p(\boldsymbol{a}, m) \leq p(\boldsymbol{a}, m+1) \leq \mathrm{card}\,\mathcal{A} \cdot p(\boldsymbol{a}, m)$, *for every integer* $m \geq 0$*;*
- $p(\boldsymbol{a}, m+n) \leq p(\boldsymbol{a}, m)p(\boldsymbol{a}, n)$, *for all integers* $m, n \geq 0$.

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a power of a prime number p.

In this paper, we are interested in Laurent series with coefficients in $\mathbb{F}_q$. Let $n_0 \in \mathbb{N}$ and consider the Laurent series:

$$f(T) = \sum_{n=-n_0}^{+\infty} a_n T^{-n} \in \mathbb{F}_q((T^{-1})).$$

Let m be a nonnegative integer. We define *the complexity* of f, denoted by $p(f, m)$, as being equal to the complexity of the infinite word $\mathbf{a} = (a_n)_{n \geq 0}$.

2.2. **Topological entropy.** Let $\mathbf{a}$ be an infinite word over an alphabet $\mathcal{A}$. *The (topological) entropy* of $\mathbf{a}$ is defined as follows:

$$h(\mathbf{a}) = \lim_{m\to\infty} \frac{\log p(\mathbf{a}, m)}{m}.$$

The limit exists as an easy consequence of the following property: $p(\mathbf{a}, n+m) \leq p(\mathbf{a}, n)p(\mathbf{a}, m)$, for every $m, n \geq 0$ (which is the second part of the Lemma 2.1). If the base of the logarithm is the cardinality of the alphabet then:

$$0 \leq h(\mathbf{a}) \leq 1.$$

Notice that, by definition, the "simpler" the sequence is, the smaller its entropy is.

Let $n_0 \in \mathbb{N}$ and consider the Laurent series

$$f(T) = \sum_{n=-n_0}^{+\infty} a_n T^{-n} \in \mathbb{F}_q((T^{-1})).$$

We define *the entropy* of f, denoted by $h(f)$, as being equal to the entropy of the infinite word $\mathbf{a} = (a_n)_{n\geq 0}$.

2.3. **Morphisms.** Let $\mathcal{A}$ (respectively $\mathcal{B}$) be an alphabet and let $\mathcal{A}^*$ (respectively $\mathcal{B}^*$) be the corresponding free monoid. A morphism σ is a map from $\mathcal{A}^*$ to $\mathcal{B}^*$ such that $\sigma(UV) = \sigma(U)\sigma(V)$ for all words $U, V \in \mathcal{A}^*$. Since the concatenation is preserved, it is then possible to define a morphism on $\mathcal{A}$.

If $\mathcal{A} = \mathcal{B}$ we can iterate the application of σ. Hence, if $a \in \mathcal{A}$, $\sigma^0(a) = a$, $\sigma^i(a) = \sigma(\sigma^{i-1}(a))$, for every $i \geq 1$.

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a morphism. The set $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ is endowed with a natural topology. Roughly, two words are close if they have a long common prefix. We can thus extend the action of a morphism by continuity to $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$. Then, a word $\mathbf{a} \in \mathcal{A}^{\mathbb{N}}$ is *a fixed point* of a morphism σ if $\sigma(\mathbf{a}) = \mathbf{a}$.

A morphism σ is *prolongable* on $a \in \mathcal{A}$ if $\sigma(a) = ax$, for some $x \in \mathcal{A}^+ := \mathcal{A}^* \backslash \{\varepsilon\}$. If σ is prolongable then the sequence $(\sigma^i(a))_{i\geq 0}$ converges to the infinite word: $\sigma^\infty(a) = \lim_{i\to\infty} \sigma^i(a) = ax\sigma(x)\sigma^2(x)\sigma^3(x)\cdots$.

Example 2.1. The Fibonacci word $\mathbf{f} = 0100101001001\cdots$ is an example of an infinite word generated by iterating the morphism: $\sigma(0) = 01$ and $\sigma(1) = 0$. More precisely, $\mathbf{f} = \sigma^\infty(0)$ is the unique fixed point of σ.

The *order of growth* of a letter x is the function $|\sigma^n(x)|$, for $n \geq 0$. In general, this function is bounded or, if not, is growing asymptotically like the function $n^{a_x} b_x^n$. A morphism is said to be *polynomially diverging* if there exists $b > 1$ such that, for any letter x, the order of growth of x is $n^{a_x} b^n$ and $a_x \geq 1$ for some x. A morphism is *exponentially diverging* if every letter x

has the order of growth $n^{a_x} b_x^n$ with $b_x > 1$ and not all b_x are equal. For more details the reader may refer to [30].

A morphism σ is said to be uniform of length $m \geq 2$ if $|g(x)| = m$. Notice that a word generated by an uniform morphism of length m is m-automatic (see for example [20]). In particular, its complexity is $O(1)$ if the word is eventually periodic; otherwise, it is $\Theta(m)$.

For more about the complexity function of words generated by morphisms there is a classical theorem of Pansiot [30] that characterizes the asymptotic behavior of factor complexity of words obtained by iterating a morphism.

2.4. **Algebraic Laurent series.** A Laurent series $f(T) = \sum_{n \geq -n_0} a_n T^{-n} \in \mathbb{F}_q((T^{-1}))$ is said to be *algebraic* over the field $\mathbb{F}_q(T)$ if there exist an integer $d \geq 1$ and polynomials $A_0(T), A_1(T), \ldots, A_d(T)$, with coefficients in $\mathbb{F}_q$ and not all zero, such that:

$$A_0 + A_1 f + \cdots + A_d f^d = 0.$$

Otherwise, f is transcendental over $\mathbb{F}_q(T)$.

Let us now give an example of Laurent series algebraic over the field of rational functions.

Example 2.2. Let us consider the formal series $f(T) = \sum_{n \geq 0} c_n T^{-n} \in \mathbb{F}_3[[T^{-1}]]$ where $\mathbf{c} := (c_n)_{n \geq 0}$ is the Cantor sequence defined as follows:

$$c_n = \begin{cases} 1 \text{ if } (n)_3 \text{ contains only } 0 \text{ and } 2; \\ 0 \text{ if } (n)_3 \text{ contains the letter } 1. \end{cases}.$$

Here $(n)_3$ denotes the expansion in base 3 of n. By definition, we get that $c_{3n} = c_n = c_{3n+2}$ and $c_{3n+1} = 0$, for all $n \in \mathbb{N}$.

We have:

$$\begin{aligned} f(T) &= \sum_{n \geq 0} c_{3n} T^{-3n} + \sum_{n \geq 0} c_{3n+1} T^{-3n-1} + \sum_{n \geq 0} c_{3n+2} T^{-3n-2} \\ &= \sum_{n \geq 0} c_n T^{-3n} + \sum_{n \geq 0} c_n T^{-3n-2}. \end{aligned}$$

Hence,

$$f(T) = f(T^3) + T^{-2} f(T^3)$$

and, since we are in characteristic 3, we obtain that f satisfies the following equation:

$$(1 + T^2) f^2(T) - T^2 = 0.$$

Thus f is an algebraic Laurent series.

Notice also that, the infinite sequence $\mathbf{c}$ is 3-automatic, as predicted by Christol's theorem, and in particular the complexity of $\mathbf{c}$ satisfies:

$$p(\mathbf{c}, m) = O(m).$$

3. An analogue of Π

In 1935, Carlitz [21] introduced for function fields in positive characteristic an analog of the exponential function defined over $\mathcal{C}_\infty$, which is the completion of the algebraic closure of $\mathbb{F}_q((T^{-1}))$ (this is the natural analogue of the complex numbers field). In order to get good properties in parallel with the complex exponential, the resulting analogue, $z \to e_C(z)$, satisfies:

$$e_C(0) = 0,\ d/dz(e_C(z)) = 1 \text{ and } e_C(Tz) = Te_C(z) + e_C(z)^q.$$

This is what we call the Carlitz exponential and the action $u \to Tu + u^q$ leads to the definition of the Carlitz $\mathbb{F}_q[T]$-module, which is in fact a particular case of Drinfeld module. The Carlitz exponential, $e_C(z)$, may be defined by the following infinite product:

$$e_C(z) = z \prod_{a\in\mathbb{F}_q[T],\, a\neq 0} (1 - \frac{z}{a\widetilde{\Pi}_q})$$

where

$$\widetilde{\Pi}_q = (-T)^{\frac{q}{q-1}} \prod_{j=1}^{\infty} \left(1 - \frac{1}{T^{q^j-1}}\right)^{-1}.$$

Since $e^z = 1$ if and only if $z \in 2\pi i\mathbb{Z}$ and since $e_C(z)$ was constructed by analogy such that $e_C(z) = 0$ if and only if $z \in \widetilde{\Pi}_q\mathbb{F}_q[T]$ (in other words the kernel of $e_C(z)$ is $\widetilde{\Pi}_q\mathbb{F}_q[T]$), we get a good analogue $\widetilde{\Pi}_q$ of $2\pi i$. In order to obtain a good analogue of the real number π, we take its one unit part and hence we obtain:

$$\Pi_q = \prod_{j=1}^{\infty} \left(1 - \frac{1}{T^{q^j-1}}\right)^{-1}.$$

For more details about analogs given by the theory of Carlitz modules, and in particular about the exponential function or its fundamental period $\widetilde{\Pi}_q$, we refer the reader to the monographs [26, 33].

If we look for the Laurent series expansion of Π_q, then we obtain that

$$\Pi_q = \prod_{j=1}^{\infty} \left(1 - \frac{1}{T^{q^j-1}}\right)^{-1} = \sum_{n\geq 0} a_n T^{-n},$$

where a_n is defined as the number of partitions of n whose parts take values in $I = \{q^j - 1, j \geq 1\}$, taken modulo p.

To compute the complexity of Π_q, we would like to find a closed formula or some recurrence relations for the sequence of partitions $(a_n)_{n\geq 0}$. This question seems quite difficult and we are not able to solve it at this moment.

However, it was shown in [6] that the inverse of Π_q has the following simple Laurent series expansion:

$$\frac{1}{\Pi_q} = \prod_{j=1}^{\infty}\left(1 - \frac{1}{X^{q^j-1}}\right) = \sum_{n=0}^{\infty} p_n X^{-n}$$

where the sequence $\mathfrak{p}_q = (p(n))_{n\geq 0}$ is defined as follows:
(2)

$$p_n = \begin{cases} 1 \text{ if } n = 0; \\ (-1)^{\operatorname{card} J} \text{ if there exists a set } J \subset \mathbb{N}^* \text{ such that } n = \sum_{j\in J}(q^j - 1); \\ 0 \text{ if there is no set } J \subset \mathbb{N}^* \text{ such that } n = \sum_{j\in J}(q^j - 1). \end{cases}$$

We mention that if such a decomposition exists, it is unique.

In the rest of this section we will prove Theorem 1.2 and 1.3.

3.1. **Proof of Theorem 1.2.** In this part we study the sequence $\mathfrak{p}_2 = (p_n^{(2)})_{n\geq 0}$, defined by the formula (2) in the case where $q = 2$. More precisely:
(3)

$$p_n^{(2)} = \begin{cases} 1 \text{ if } n = 0 \text{ or if there exists } J \subset \mathbb{N}^* \text{ such that } n = \sum_{j\in J}(2^j - 1); \\ 0 \text{ otherwise.} \end{cases}$$

In order to lighten the notations, in the rest of this subsection we set $p_n := p_n^{(2)}$ so that $\mathfrak{p}_2 = p_0p_1p_2\cdots$.

For every $n \geq 1$, we denote by W_n the factor of $\mathfrak{p}_2$ that occurs between positions $2^n - 1$ and $2^{n+1} - 2$, that is:

$$W_n := p_{2^n-1}\cdots p_{2^{n+1}-2}.$$

We also set $W_0 := 1$. Observe that $|W_n| = 2^n$.
With these notations the infinite word $\mathfrak{p}_2$ can be factorized as:

$$\mathfrak{p}_2 = \underbrace{1}_{W_0}\underbrace{10}_{W_1}\underbrace{1100}_{W_2}\underbrace{11011000}_{W_3}\cdots = W_0W_1W_2\cdots.$$

In order to compute the complexity function of $\mathfrak{p}_2$, we need the following lemmas.

Lemma 3.1. *Let n and k be two positive integers such that: $k < 2^n - 1$. Then k can be written as $\sum_{j\in J}(2^j - 1)$ if and only if $k + (2^n - 1)$ can be written as $\sum_{i\in I}(2^i - 1)$, where I and J are finite subsets of $\mathbb{N}^*$.*

Remark 3.1. This is equivalent to say that $a_k = 1$ if and only if $a_{k+(2^n-1)} = 1$.

Proof. The first part is pretty obvious. If $k = \sum_{j\in J}(2^j - 1)$, then:

$$k + (2^n - 1) = \sum_{j\in J}(2^j - 1) + (2^n - 1) = \sum_{j\in J\cup\{n\}}(2^j - 1).$$

For the second part, let suppose that $k + (2^n - 1) = \sum_{i\in I}(2^i - 1)$. We prove that k can be also represented in this form. More precisely, we show that $n \in I$ and consequently $k = \sum_{i\in I\setminus\{n\}}(2^i - 1)$.

Notice that I cannot contain any index greater than n since $k < 2^n - 1$. We argue by contradiction and we assume that I does not contain n. Then $\sum_{i \in I} (2^i - 1) < 2^n - 1$ since

$$\sum_{i \in I} (2^i - 1) \leq \sum_{i=1}^{n-1} (2^i - 1) = 2(2^{n-1} - 1) - (n-1) = 2^n - n - 1 < 2^n - 1.$$

Hence, if n does not belong to I, then $k + (2^n - 1) < 2^n - 1$ which is absurd. Consequently, n belongs to I and thus $k = \sum_{i \in I \setminus \{n\}} (2^i - 1)$. $\square$

Lemma 3.2. *For every $n \geq 2$ we have $W_n = 1W_1W_2 \cdots W_{n-1}0$.*

Proof. Clearly, the word W_n begins with 1 because $p_{2^n-1} = 1$. The word W_n ends with 0 since the last letter is $p_{2^{n+1}-2} = 0$. To explain the structure of W_n, we split the word W_n into subwords as follows:

$$W_n = \underbrace{p_{2^n-1}}_{1} \underbrace{p_{(2^n-1)+(2-1)} p_{(2^n-1)+(2^2-2)}}_{W_1'} \underbrace{p_{(2^n-1)+(2^2-1)} \cdots p_{(2^n-1)+(2^3-2)}}_{W_2'}$$
$$\cdots \underbrace{p_{(2^n-1)+(2^{n-1}-1)} \cdots p_{(2^n-1)+(2^n-2)}}_{W_{n-1}'} \underbrace{p_{2^{n+1}-2}}_{0}.$$

Since by Lemma 3.1 $p_{(2^n-1)+k} = p_k$ for $k < 2^n - 1$, we obtain that $W_i' = W_i$, for $1 \leq i \leq n-1$. $\square$

Since the subword W_n ends with 0, we can define U_n by $W_n := U_n0$, for every $n \geq 1$. Thus, $U_1 = 1$, $U_2 = 110$.

Lemma 3.3. *For every $n \geq 1$, we have $U_{n+1} = U_nU_n0$.*

Proof. By Lemma 3.2, $U_n = 1W_1W_2 \cdots W_{n-1}$ for all $n \geq 2$. Consequently:

$$U_{n+1} = \underbrace{1W_1W_2 \cdots W_{n-1}}_{U_n} W_n = U_nW_n = U_n \underbrace{U_n0}_{W_n}.$$

$\square$

Lemma 3.4. *For every $n \geq 2$, there exists a word Z_n such that $W_n = 1Z_n10^n$ and $0^n \ntrianglelefteq Z_n$ (in other words W_n ends with exactly n zeros and Z_n does not contain blocks of 0 of length larger than $n-1$). This is equivalent to say that $U_n = 1Z_n10^{n-1}$ and $0^n \ntrianglelefteq Z_n$.*

Proof. We argue by induction on n.

For $n \geq 2$, $W_2 = 1100$ ends with two zeros and obviously there are no other zeros.

We assume that W_n ends with n zeros and does not contain other block of zeros of length greater than $n-1$. We show this statement holds for $n+1$. By Lemma 3.3

$$W_{n+1} = U_{n+1}0 = U_nU_n00.$$

As U_n ends exactly with $n-1$ zeros (by induction hypothesis), then W_{n+1} ends also by $n+1$ zeros. Since we have that $U_n = 1Z_n10^{n-1}$ and $0^n \ntrianglelefteq Z_n$ then $W_{n+1} = 1Z_n10^{n-1}1Z_n10^{n-1}00 = 1Z_{n+1}0^{n+1}$, when $Z_{n+1} := Z_n10^{n-1}1Z_n$. Since $0^n \ntrianglelefteq Z_n$, then $0^{n+1} \ntrianglelefteq Z_{n+1}$. This completes the proof. □

Lemma 3.5. *For every $n \geq 1$, let $A_n := \{U_n^2 0^k,\ k \geq 1\}$. Then $\mathfrak{p}_2 \in A_n^{\mathbb{N}}$.*

Proof. Let $n \geq 1$. By definition of W_n and U_n and by Lemma 3.2, the infinite word $\mathfrak{p}_2$ can be factorized as:

$$\mathfrak{p}_2 = \underbrace{1W_1W_2\cdots W_{n-1}}_{U_n}\underbrace{U_n0}_{W_n}\underbrace{U_{n+1}0}_{W_{n+1}}\underbrace{U_{n+2}0}_{W_{n+2}}\cdots . \tag{4}$$

We prove that for every positive integer k, there exist a positive integer r and $k_1, k_2, \ldots, k_r \in \mathbb{N}^*$ such that:

$$U_{n+k} = U_n^2 0^{k_1} U_n^2 0^{k_2} \cdots U_n^2 0^{k_r}. \tag{5}$$

We argue by induction on k. For $k = 1$, we have $U_{n+1} = U_nU_n0 = U_n^2 0$. We suppose that the relation (5) is true for k and we show it for $k+1$. By Lemma 3.3:

$$U_{n+k+1} = U_{n+k}U_{n+k}0 = U_n^2 0^{k_1} U_n^2 0^{k_2} \cdots U_n^2 0^{k_r} U_n^2 0^{k_1} U_n^2 0^{k_2} \cdots U_n^2 0^{k_r+1}.$$

By equality (4), this ends the proof. □

Fix $m \in \mathbb{N}$. Then, there is a unique integer n such that:

$$2^{n-1} < m \leq 2^n. \tag{6}$$

Lemma 3.6. *Let $m \in \mathbb{N}$. All distinct words of length m of $\mathfrak{p}_2$ occur in the prefix:*

$$P_m = W_0W_1\cdots W_m.$$

Proof. Let m, n be some positive integer satisfying 6.

We show that all distinct words of length m occur in the prefix

$$P_m = W_0W_1W_2\cdots W_{n-1}W_n\cdots W_m = U_n\underbrace{U_n0}_{W_n}\underbrace{U_nU_n00}_{W_{n+1}}\underbrace{U_nU_n0U_nU_n000}_{W_{n+2}}\cdots W_m;$$

the second identity follows by Lemmas 3.2 and 3.3.

Notice that we have to consider all the words till W_m because the word 0^m first occurs in W_m.

Also, by Lemma 3.4 and using the identity (5), W_i ends with $U_nU_n0^{i-n+1}$, for every $i \geq n+1$. Consequently, all the words $U_nU_n0^k$, $0 \leq k \leq m-n+1$ are factors of P_m.

Moreover, notice that if $B_n := \{U_nU_n0^k, 0 \leq k \leq m-n+1\}$, then $P_m \in B_n^*$. This follows from Lemmas 3.5 and 3.4 (there are no blocks of zeros of length greater than m in $W_0W_1\cdots W_m$).

After the occurrence of W_m, it is not possible to see new different subwords of length m. Indeed, suppose that there exists a word F of length m that occur in $W_{m+1}W_{m+2}W_{m+3}\cdots$ and does not occur in P_m. Then, by Lemma 3.5 and by the remark above, F must occur in the words $U_n0^kU_n$, with $k \geq m-n+1$. But since U_n ends with $n-1$ zeros (by Lemma 3.4), F must be equal to 0^m or 0^iP_i, where $i \geq m-n+2$ and $P_i \prec_p U_n$, or F must occur in U_n. But all these words already occur in P_m. This contradicts our assumption. $\square$

3.1.1. *An upper bound for* $p(\frac{1}{\Pi_2}, m)$. In this part we prove the following result.

Proposition 3.1.

$$p(\mathfrak{p}_2, m) \leq \frac{(m-\log m)(m+\log m+2)}{2} + 2m.$$

Proof. In order to find all different factors of length m that occur in $\mathfrak{p}_2$, it suffices, by Lemmas 3.5 and 3.6, to consider factors appearing in the word U_nU_n and in the sets $i(U_n, 0^k, U_n)$, where $1 \leq k \leq m-n$.

In the word U_nU_n we can find at most $|U_n|$ distinct words of length m. Since $|U_n| = 2^n - 1$ and $2^{n-1} < m \leq 2^n$, the number of factors of length m that occur in U_nU_n is at most 2^n, so at most $2m$.

Also, it is not difficult to see that $\left|i(U_n, 0^k, U_n)\right| \cap \mathcal{A}^m \leq m-k+1$. The total number of subwords occurring in all these sets, for $1 \leq k \leq m-n$, is less than or equal to:

$$\sum_{k=1}^{m-n}(m-k+1) = (m+1)(m-n) - \frac{(m-n)(m-n+1)}{2}.$$

Counting all these words and using the fact that $2^{n-1} < m \leq 2^n$, we obtain that:

$$p(\mathfrak{p}_2, m) \leq 2m + \frac{(m-n)(m+n+1)}{2} < \frac{(m-\log m)(m+\log m+2)}{2} + 2m$$

as claimed. $\square$

3.1.2. *A lower bound for* $p(\frac{1}{\Pi_2}, m)$. In this part we prove the following result.

Proposition 3.2.

$$p(\mathfrak{p}_2, m) \geq \frac{(m-\log m)(m-\log m+1)}{2}.$$

Proof. By Lemma 3.6, we have to look for distinct words of length m occurring in $W_nW_{n+1}\cdots W_m$.

In order to prove this proposition, we use the final blocks of 0 from each W_i. These blocks are increasing (as we have shown in Lemma 3.4). First, in

the word W_m we find for the first time the word of length m: 0^m.

In the set $i(W_{m-1}, \varepsilon, W_m)$, we find two distinct words of length m that cannot be seen before (10^{m-1} and $0^{m-1}1$) since there are no other words containing blocks of zeros of length $m-1$ in $i(W_k, \varepsilon, W_{k+1})$, for $k < m-1$.

More generally, fix k such that $n \leq k \leq m-2$. Since

$$W_k W_{k+1} = \underbrace{1Z_k 10^k}_{W_k} \underbrace{1Z_{k+1}10^{k+1}}_{W_{k+1}},$$

in $i(W_k, \varepsilon, W_{k+1})$ we find $m-k+1$ words of length m of form $\alpha_k 0^k \beta_k$. More precisely, the words we count here are the following: $S_{m-k-1}10^k$, $S_{m-k-2}10^k1$, $S_{m-k-3}10^k1T_1$,..., $S_1 0^k 1T_{m-k-2}$, 0^k1T_{m-k-1}, where $S_i \prec_s Z_k$ and $T_i \prec_p Z_{k+1}$, $|S_i| = |T_i| = i$, for every integer i, $1 \leq i \leq m-k-1$.

All these words cannot be seen before, that is in $i(W_s, \varepsilon, W_{s+1})$, for $s < k$, since there are no blocks of zeros of length k before the word W_k (according to Lemma 3.4). Also, in $i(W_s, \varepsilon, W_{s+1})$, for $s > k$, we focus on the words $\alpha_s 0^s \beta_s$ and hence they are different from all the words seen before (because $k < s$).

Consequently, the total number of subwords of length m of form $\alpha_k 0^k \beta_k$ considered before, is equal to

$$1 + 2 + \ldots + (m-n+1) = \frac{(m-n+1)(m-n+2)}{2}.$$

Since $2^{n-1} < m \leq 2^n$ we obtain the desired lower bound. $\square$

Proof of Theorem 1.2. It follows from Propositions 3.1 and 3.2. $\square$

A consequence of Theorem 1.2 and Theorem 1.1 is the following result of transcendence.

Corollary 3.1. *Let K be a finite field and $(p_n^{(2)})_{n\geq 0}$ the sequence defined in (3). Let us consider the associated formal series over K:*

$$f(T) := \sum_{n\geq 0} p_n^{(2)} T^{-n} \in K[[[T^{-1}]].$$

Then f is transcendental over $K(T)$.

Notice that, if $K = \mathbb{F}_2$ then the formal series f coincide with $1/\Pi_2$ and hence Corollary 3.1 implies Corollary 1.1.

Remark 3.2. In [7], the authors proved that the sequence $\mathfrak{p}_2$ is the fixed point of the morphism σ defined by $\sigma(1) = 110$ and $\sigma(0) = 0$.

In an unpublished note [5], Allouche showed that the complexity of the sequence $\mathfrak{p}_2$ satisfies, for all $m \geq 1$, the following inequality:

$$p(\mathfrak{p}_2, m) \geq Cm \log m,$$

for some strictly positive constant C.

3.2. **Proof of Theorem 1.3.** In this part we study the sequence $\mathfrak{p}_q = (p_n^{(q)})_{n\geq 0}$ defined by the formula (2) in the case where $q \geq 3$. In the following, we will consider the case $q = p^n$, where $p \geq 3$.

Proposition 3.3. *Let $q \geq 3$. For every positive integer m:*

$$p(\mathfrak{p}_q, m) \leq (2q + 4)m + 2q - 3.$$

In particular, this proves the Theorem 1.3. Indeed, we do not have to find a lower bound for the complexity function, as the sequence $\mathfrak{p}_q$ is not eventually periodic (see Remark 3.5) and thus, by the inequality (1) we have that:

$$p(\mathfrak{p}_q, m) \geq m + 1,$$

for any $m \geq 0$.

In order to lighten the notations, we set in the sequel $p_n := p_n^{(q)}$ so that $\mathfrak{p}_q = p_0 p_1 p_2 \cdots$.

For every $n \geq 1$, we denote by W_n the factor of $\mathfrak{p}_q$ defined in the following manner:

$$W_n := p_{q^n-1} \cdots p_{q^{n+1}-2}.$$

Let us fix $W_0 := 0^{q-2} = \underbrace{00\cdots 0}_{q-2}$ and $\alpha_0 := q - 2$. Thus $W_0 = 0^{\alpha_0}$.

In other words, W_n is the factor of $\mathfrak{p}_q$ occurring between positions $q^n - 1$ and $q^{n+1} - 2$. Notice that $|W_n| = q^n(q - 1)$.

With these notations the infinite word $\mathfrak{p}_q$ may be factorized as follows:

$$\mathfrak{p}_q = 1\underbrace{00\cdots 0}_{W_0}\underbrace{(-1)00\cdots 0}_{W_1}\underbrace{(-1)\cdots 00100\cdots 0}_{W_2}(-1)00\cdots.$$

In the following, we prove some lemmas that we use in order to bound from above the complexity function of $\mathfrak{p}_q$.

Lemma 3.7. *Let k and n be two positive integers such that:*

$$k \in \left[2(q^n - 1), q^{n+1} - 2\right].$$

Then there is no set $J \subset \mathbb{N}^*$ such that $k = \sum_{j\in J}(q^j - 1)$. In other words, $p_k = 0$.

Proof. We argue by contradiction and we assume that there exists a set J such that $k = \sum_{j \in J} (q^j - 1)$. Since $k < q^{n+1} - 1$, then obviously J must be a subset of $\{1, 2, 3, \dots, n\}$. Consequently

$$k = \sum_{j \in J} (q^j - 1) \leq \sum_{j=1}^{n} (q^j - 1).$$

Then

$$k \leq \sum_{j=1}^{n} (q^j - 1) = q\frac{q^n - 1}{q - 1} - n < 2(q^n - 1) < k$$

which is absurd. □

Lemma 3.8. *Let k and n be two positive integers such that: $k < q^n - 1$. Then k can be written as $\sum_{j \in J} (q^j - 1)$ if and only if $k + (q^n - 1)$ can be written as $\sum_{j \in I} (q^j - 1)$, where I and J are finite subsets of $\mathbb{N}^*$. Moreover, $J \cup \{n\} = I$.*

Remark 3.3. This is equivalent to say that $p_k = -p_{k+(q^n-1)}$ for k and n two positive integers such that $k < q^n - 1$.

Proof. The proof is similar to that of Lemma 3.1 (just replace 2 by q). □

If $W = a_1 a_2 \cdots a_l \in \{0, 1, -1\}^l$ then set $\widehat{W} := (-a_1)(-a_2)\cdots(-a_l)$.

Lemma 3.9. *For every $n \geq 1$ we have the following:*

$$W_n = (-1)\widehat{W_0}\widehat{W_1}\cdots\widehat{W_{n-1}}0^{\alpha_n}$$

with $\alpha_n = (q^{n+1} - 1) - 2(q^n - 1)$.

Proof. Obviously, the word W_n begins with -1 since $p_{q^n-1} = -1$. In order to prove the relation above it suffices to split W_n into subwords as follows:

$$W_n = \underbrace{p_{q^n-1}}_{-1}\underbrace{0\cdots 0}_{W_0'}\underbrace{p_{(q^n-1)+(q-1)}p_{(q^n-1)+(q^2-2)}}_{W_1'}\underbrace{p_{(q^n-1)+(q^2-1)}\cdots p_{(q^n-1)+(q^3-2)}}_{W_2'}$$

$$\cdots\underbrace{p_{(q^n-1)+(q^{n-1}-1)}\cdots p_{(q^n-1)+(q^n-2)}}_{W_{n-1}'}\underbrace{p_{2(q^n-1)}\cdots p_{q^{n+1}-2}}_{0^{\alpha_n}}.$$

Since $p_{(q^n-1)+k} = -p_k$, for every $k < q^n - 1$ (by Lemma 3.8), we obtain that $W_i' = \widehat{W_i}$, for $0 \leq i \leq n - 1$. Lemma 3.10 ends the proof. □

Since the subword W_n ends with 0^{α_n}, we can define U_n as prefix of W_n such that $W_n := U_n 0^{\alpha_n}$, for every $n \geq 1$. Notice that $|U_n| = q^n - 1$.

Lemma 3.10. *For every $n \geq 1$, we have $U_{n+1} = U_n\widehat{U_n}0^{\alpha_n}$.*

Proof. By Lemma 3.9, $U_n = (-1)\widehat{W_0}\widehat{W_1}\cdots\widehat{W_{n-1}}$. Consequently:

$$U_{n+1} = \underbrace{(-1)\widehat{W_0}\widehat{W_1}\cdots\widehat{W_{n-1}}}_{U_n}\widehat{W_n} = U_n\widehat{U_n 0^{\alpha_n}} = U_n\widehat{U_n}0^{\alpha_n}.$$

□

Remark 3.4. Since $q \geq 3$ we have $\alpha_n \geq |U_n|$ for every $n \geq 1$. Moreover $(\alpha_n)_{n\geq 1}$ is a positive and increasing sequence.

Lemma 3.11. *For every $n \geq 1$, let $A_n := \{U_n, \widehat{U_n}, 0^{\alpha_i}, i \geq n\}$. Then $\mathfrak{p}_q \in A_n^{\mathbb{N}}$.*

Proof. Let $n \geq 1$. By definition of W_n and U_n, the infinite word $\mathfrak{p}_q$ can be factorized as:

$$\mathfrak{p}_q = \underbrace{1W_0W_1\cdots W_{n-1}}_{V_n}W_nW_{n+1}\cdots.$$

By Lemma 3.9, since $U_n = (-1)\widehat{W_0}\widehat{W_1}\cdots\widehat{W_{n-1}}$ then the prefix $V_n = \widehat{U_n}$.

Also, $W_{n+1} = U_n\widehat{U_n}0^{\alpha_n}0^{\alpha_{n+1}}$, $W_{n+2} = U_n\widehat{U_n}0^{\alpha_n}\widehat{U_n}U_n0^{\alpha_n+\alpha_{n+1}+\alpha_{n+2}}$. Keeping on this procedure, W_n can be written as a concatenation of U_n, $\widehat{U_n}$ and 0^{α_i}, $i \geq n$. More precisely, $\mathfrak{p}_q$ can be written in the following manner:

$$\mathfrak{p}_q = \widehat{U_n}\underbrace{U_n0^{\alpha_n}}_{W_n}\underbrace{U_n\widehat{U_n}0^{\alpha_n+\alpha_{n+1}}}_{W_{n+1}}\underbrace{U_n\widehat{U_n}0^{\alpha_n}\widehat{U_n}U_n0^{\alpha_n+\alpha_{n+1}+\alpha_{n+2}}}_{W_{n+2}}\cdots.$$

□

Proof of Proposition 3.3. Let $m \in \mathbb{N}$. Then there exists a unique positive integer n, such that:

$$q^{n-1} - 1 \leq m < q^n - 1.$$

By Lemma 3.11 and the Remark 3.4, between the words U_n and $\widehat{U_n}$ (when they do not occur consecutively), there are only blocks of zeros of length greater than $\alpha_n \geq |U_n| = q^n - 1$ and thus greater than m. Hence, all distinct factors of length m appear in the following words: $U_n\widehat{U_n}$, $\widehat{U_n}U_n$, $0^{\alpha_n}U_n$, $0^{\alpha_n}\widehat{U_n}$, $U_n0^{\alpha_n}$ and $\widehat{U_n}0^{\alpha_n}$.

In $U_n\widehat{U_n}$ we may find at most $|U_n\widehat{U_n}| - m + 1 = 2|U_n| - m + 1$ factors at length m. In $\widehat{U_n}U_n$ we may find at most $m-1$ new different factors of length m. More precisely, they form the set $i(\widehat{U_n}, \varepsilon, U_n)^+$.

In $0^{\alpha_n}U_n$ (respectively $0^{\alpha_n}\widehat{U_n}$, $U_n0^{\alpha_n}$, $\widehat{U_n}0^{\alpha_n}$) we may find at most m (respectively $m-1$) new different factors (they belong to $i(0^{\alpha_n}, \varepsilon, U_n)^+ \cup \{0^m\}$, respectively $i(0^{\alpha_n}, \varepsilon, \widehat{U_n})^+$, $i(U_n, \varepsilon, 0^{\alpha_n})^+$ and $i(\widehat{U_n}, \varepsilon, 0^{\alpha_n})^+$).

Consequently, the number of such subwords is at most $2|U_n| + 4m - 3$. Since $U_n = q^n - 1 = q(q^{n-1} - 1) + q - 1 \leq qm + q - 1$ we obtain that:

$$p(\mathfrak{p}_q, m) \leq 2(qm + q - 1) + 4m - 3 \leq (2q + 4)m + 2q - 3.$$

□

Remark 3.5. It is not difficult to prove that $\mathfrak{p}_q$ is not eventually periodic. Indeed, recall that $\mathfrak{p}_q = W_0 W_1 W_2 \cdots$. Using Theorem 3.9 and the Remark 3.4,

$$\mathfrak{p}_q = A_1 0^{l_1} A_2 0^{l_2} \cdots A_i 0^{l_i} \cdots , \tag{7}$$

where A_i, $i \geq 1$, are finite words such that $A_i \neq 0^{|A_i|}$ and $(l_i)_{i \geq 1}$ is a strictly increasing sequence.

Remark 3.6. This part concerns the case where $q \geq 3$. If the characteristic of the field is 2, that is, if $q = 2^n$, where $n \geq 2$, then, in the proof we have that $-1 = 1$, but the structure of $\mathfrak{p}_q$ remain the same. We will have certainly a lower complexity, but $\mathfrak{p}_q$ is still on the form (7), and thus $p(\mathfrak{p}_q, m) \leq (2q + 4)m + 2q - 3$.

4. Closure properties of two classes of Laurent series

It is natural to classify Laurent series in function of their complexity. In this section we study some closure properties for the following classes:

$$\mathcal{P} = \{f \in \mathbb{F}_q((T^{-1})), \text{there exists } K \text{ such that } p(f, m) = O(m^K)\}$$

and, more generally,

$$\mathcal{Z} = \{f \in \mathbb{F}_q((T^{-1})), \text{ such that } \ h(f) = 0\}.$$

Clearly, $\mathcal{P} \subset \mathcal{Z}$. We recall that h is the topological entropy defined in Section 2.

We have already seen, in Theorem 1.1, that the algebraic Laurent series belong to $\mathcal{P}$ and $\mathcal{Z}$. Also, by Theorem 1.2 and 1.3, $\frac{1}{\Pi_q}$ belongs to $\mathcal{P}$. Hence, $\mathcal{P}$, and more generally $\mathcal{Z}$, seem to be two important objects of interest for this classification.

The main result we will prove in this section is Theorem 1.4.

In the second part, we will prove the stability of $\mathcal{P}$ and $\mathcal{Z}$ under Hadamard product, formal derivative and Cartier operator.

4.1. **Proof of Theorem 1.4.** The proof of Theorem 1.4 is a straightforward consequence of Propositions 4.1 and 4.3 below.

Proposition 4.1. *Let f and g be two Laurent series belonging to $\mathbb{F}_q((T^{-1}))$. Then, for every integer $m \geq 1$, we have:*

$$\frac{p(f,m)}{p(g,m)} \leq p(f+g,m) \leq p(f,m)p(g,m).$$

Proof. Let $f(T) := \sum_{i\geq -i_1} a_i T^{-i}$ and $g(T) := \sum_{i\geq -i_2} b_i T^{-i}$, $i_1, i_2 \in \mathbb{N}$.

By definition of the complexity of Laurent series (see Section (2.1)), for every $m \in \mathbb{N}$:

$$p(f(T)+g(T),m) = p(\sum_{i\geq 0} c_i T^{-i}, m),$$

where $c_i := (a_i + b_i) \in \mathbb{F}_q$. Thus we may suppose that

$$f(T) := \sum_{i\geq 0} a_i T^{-i} \text{ and } g(T) := \sum_{i\geq 0} b_i T^{-i}.$$

We denote by $\mathbf{a} := (a_i)_{i\geq 0}$, $\mathbf{b} := (b_i)_{i\geq 0}$ and $\mathbf{c} := (c_i)_{i\geq 0}$.

For the sake of simplicity, throughout this part, we set $x(m) := p(f,m)$ and $y(m) := p(g,m)$. Let $\mathcal{L}_{f,m} := \{U_1, U_2, \ldots, U_{x(m)}\}$ (resp. $\mathcal{L}_{g,m} := \{V_1, V_2, \ldots, V_{y(m)}\}$) be the set of different factors of length m of the sequence of coefficients of f (respectively of g). As the sequence of coefficients of the Laurent series $f+g$ is obtained by the termwise addition of the sequence of coefficients of f and the sequence of coefficients of g, we deduce that:

$$\mathcal{L}_{f+g,m} \subseteq \{U_i + V_j,\ 1 \leq i \leq x(m),\ 1 \leq j \leq y(m)\}$$

where $\mathcal{L}_{f+g,m}$ is the set of all distinct factors of length m occurring in $\mathbf{c}$, and where the sum of two words with the same length $A = a_1 \cdots a_m$ and $B = b_1 \cdots b_m$ is defined as

$$A + B = (a_1 + b_1) \cdots (a_m + b_m)$$

(each sum being considered over $\mathbb{F}_q$). Consequently, $p(f+g,m) \leq p(f,m)p(g,m)$.

We shall prove now the first inequality using Dirichlet's principle.

Notice that if $x(m) < y(m)$ the inequality is obvious.

Assume now that $x(m) \geq y(m)$. Remark that if we extract $x(m)$ subwords of length m from $\mathbf{b}$, there is at least one word which appears at least $\left\lceil \frac{x(m)}{y(m)} \right\rceil$ times.

For every fixed m, there exist exactly $x(m)$ different factors of $\mathbf{a}$. The subwords of $\mathbf{c}$ will be obtained adding factors of length m of $\mathbf{a}$ with factors of length m of $\mathbf{b}$.

Consider all distinct factors of length m of $\mathbf{a}$: $U_1, U_2, \ldots, U_{x(m)}$, that occur in positions $i_1, i_2, \ldots, i_{x_m}$. Looking in the same positions in $\mathbf{b}$, we have $x(m)$

factors of length m belonging to $\mathcal{L}_{g,m}$. Since $x(m) \geq y(m)$, by the previous remark, there is one word W which occur at least $\left\lceil \frac{x(m)}{y(m)} \right\rceil$ times in $\mathbf{b}$.

Since we have $U_i + W \neq U_j + W$ if $U_i \neq U_j$, the conclusion follows immediately. □

Remark 4.1. In fact, the first inequality may also be easily obtained from the second one, but we chose here to give a more intuitive proof. Indeed, if we denote $f := h_1 + h_2$, $g := -h_2$, where $h_1, h_2 \in \mathbb{F}_q((T^{-1}))$, the first relation follows immediately, since $p(h_2, m) = p(-h_2, m)$, for any $m \in \mathbb{N}$.

Remark 4.2. If $f \in \mathbb{F}_q((T^{-1}))$ and $a \in \mathbb{F}_q[T]$ then, obviously, there exists a constant C (depending on the degree of the polynomial a) such that, for any $m \in \mathbb{N}$,

$$p(f + a, m) \leq p(f, m) + C.$$

Remark 4.3. Related to Proposition 4.1, one can naturally ask if it is possible to saturate the inequalities in Proposition 4.1. By Remark 4.1, it suffices to show that this is possible for one inequality. In the sequel, we construct two explicit examples of Laurent series of linear complexity such that their sum has quadratic complexity.

Let α and β be two irrational numbers such that 1, α and β are linearly independent over $\mathbb{Q}$. For any $i \in \{\alpha, \beta\}$ we consider the following rotations:

$$R_i : \mathbb{T}^1 \to \mathbb{T}^1 \qquad x \to \{x + i\},$$

where $\mathbb{T}^1$ is the circle $\mathbb{R}/\mathbb{Z}$, identified to the interval $[0, 1)$.

We may partition $\mathbb{T}^1$ in two intervals I_i^0 and I_i^1, delimited by 0 and $1 - i$. We denote by ν_i the coding function:

$$\nu_i(x) = \begin{cases} 0 & \text{if } x \in I_i^0; \\ 1 & \text{if } x \in I_i^1. \end{cases}$$

We define $\mathbf{a} := (a_n)_{n \geq 0}$ such that, for any $n \geq 0$,

$$a_n = \nu_\alpha(R_\alpha^n(0)) = \nu_\alpha(\{n\alpha\})$$

and $\mathbf{b} := (a_n)_{n \geq 0}$ such that, for any $n \geq 0$,

$$b_n = \nu_\beta(R_\beta^n(0)) = \nu_\beta(\{n\beta\}).$$

Let us consider $f(T) = \sum_{n \geq 0} a_n T^{-n}$ and $g(T) = \sum_{n \geq 0} b_n T^{-n}$ be two elements of $\mathbb{F}_3((T^{-1}))$. We will prove that, for any $m \in \mathbb{N}$, we have:

$$p(f + g, m) = p(f, m)p(g, m). \tag{8}$$

We thus provide an example of two infinite words whose sum has a maximal complexity, in view of Proposition 4.1.

A sequence of form $(\nu(R_\alpha^n(x)))_{n \geq 0}$ is a particular case of rotation sequences. It is not difficult to see that the complexity of the sequence $\mathbf{a}$

satisfies $p(\mathbf{a}, m) = m+1$ for any $m \in \mathbb{N}$ and hence $\mathbf{a}$ is Sturmian. For a complete proof, the reader may consult the monograph [31], but also the original paper of Morse and Hedlund [28], where they prove that every Sturmian sequence is a rotation sequence.

Let $m \in \mathbb{N}$. Let

$$\mathcal{L}_{\mathbf{a},m} := \{U_1, U_2, \ldots, U_{m+1}\}$$

and respectively

$$\mathcal{L}_{\mathbf{b},m} := \{V_1, V_2, \ldots, V_{m+1}\}$$

be the set of distinct factors of length m that occur in $\mathbf{a}$, respectively in $\mathbf{b}$.

In order to prove the relation (8), we show that

$$\mathcal{L}_{\mathbf{a}+\mathbf{b},m} = \{U_i + V_j, 1 \leq i, j \leq m+1\}. \tag{9}$$

Let $I := [0, 1)$. It is well-known (see for example Proposition 6.1.7 in [31]) that, using the definition of the sequence $\mathbf{a}$ (respectively of $\mathbf{b}$), we can split I in $m+1$ intervals of positive length $J_1, J_2, \ldots, J_{m+1}$ (respectively $L_1, L_2, \ldots, L_{m+1}$) corresponding to $U_1, U_2, \ldots, U_{m+1}$ (respectively $V_1, V_2, \ldots, V_{m+1}$) such that:

$$\{n\alpha\} \in J_k \text{ if and only if } a_n a_{n+1} \cdots a_{n+m-1} = U_k$$

$$(\text{respectively } \{n\beta\} \in L_k \text{ if and only if } b_n b_{n+1} \cdots b_{n+m-1} = V_k.)$$

In other words, $\{n\alpha\} \in J_k$ (resp. $\{n\beta\} \in L_k$) if and only if the factor U_k (resp.V_k) occurs in $\mathbf{a}$ (resp. $\mathbf{b}$) at the position n.

Now we use the well-known Kronecker's theorem which asserts that the sequence of fractional parts $(\{n\alpha\}, \{n\beta\})_{n \geq 0}$ is dense in the square $[0, 1)^2$ since by assumption 1, α and β are linearly independent over $\mathbb{Q}$.

In particular, this implies that, for any pair $(i, j) \in \{0, 1, \ldots, m+1\}^2$, there exists a positive integer n such that $(\{n\alpha\}, \{n\beta\}) \in J_i \times L_k$. This is equivalently to say that, for any pair of factors $(U_i, V_j) \in \mathcal{L}_{\mathbf{a},m} \times \mathcal{L}_{\mathbf{b},m}$, there exists n such that $U_i = a_n a_{n+1} \cdots a_{n+m-1}$ and $V_k = b_n b_{n+1} \cdots b_{n+m-1}$. This proves Equality (9) and more precisely, since we are in characteristic 3, we have the following equality:

$$\operatorname{Card} \mathcal{L}_{\mathbf{a}+\mathbf{b},m} = \operatorname{Card} \mathcal{L}_{\mathbf{a},m} \cdot \operatorname{Card} \mathcal{L}_{\mathbf{b},m} = (m+1)^2.$$

We point out the following consequence of Proposition 4.1.

Corollary 4.1. *Let $f_1, f_2, \ldots, f_l \in \mathbb{F}_q((T^{-1}))$. Then for every $m \in \mathbb{N}$ and for every integer $i \in [1; l]$ we have the following:*

$$\frac{p(f_i, m)}{\prod_{j \neq i, 1 \leq j \leq l} p(f_j, m)} \leq p(f_1 + f_2 + \cdots + f_l, m) \leq \prod_{1 \leq j \leq l} p(f_j, m).$$

Notice that these inequalities can be saturated, just generalizing the construction above (choose l Sturmian sequences of irrational slopes $\alpha_1, \alpha_2, \ldots, \alpha_l$, such that $1, \alpha_1, \alpha_2, \ldots, \alpha_l$ are linearly independent over $\mathbb{Q}$).

We shall prove next that the sets $\mathcal{P}$ and $\mathcal{Z}$ are closed under multiplication by rationals. Let us begin with a particular case, that is the multiplication by a polynomial.

Proposition 4.2. *Let $b(T) \in \mathbb{F}_q[T]$ and $f(T) \in \mathbb{F}_q((T^{-1}))$. Then there is a positive constant M (depending only on $b(T)$), such that for all $m \in \mathbb{N}$:*

$$p(bf, m) \leq M\ p(f, m).$$

Proof. Let

$$b(T) := b_0 T^r + b_1 T^{r-1} + \cdots + b_r \in \mathbb{F}_q[T]$$

and

$$f(T) := \sum_{i \geq -i_0} a_i T^{-i} \in \mathbb{F}_q((T^{-1})),\ i_0 \in \mathbb{N}.$$

Then

$$\begin{aligned} b(T)f(T) &= b(T)\left(\sum_{i=-i_0}^{-1} a_i T^{-i} + \sum_{i\geq 0} a_i T^{-i}\right) \\ &= b(T)\left(\sum_{i=-i_0}^{-1} a_i T^{-i}\right) + b(T)\left(\sum_{i\geq 0} a_i T^{-i}\right). \end{aligned} \tag{10}$$

Now, the product

$$\begin{aligned} b(T)(\sum_{i\geq 0} a_i T^{-i}) &= T^r(b_0 + b_1 T^{-1} + b_2 T^{-2} + \cdots + b_r T^{-r})(\sum_{i\geq 0} a_i T^{-i}) \\ &:= T^r(\sum_{j\geq 0} c_j T^{-j}) \end{aligned} \tag{11}$$

where the sequence $\mathbf{c} := (c_j)_{j\geq 0}$ is defined as follows:

$$c_j = \begin{cases} b_0 a_j + b_1 a_{j-1} + \cdots + b_j a_0 \text{ if } j < r \\ b_0 a_j + b_1 a_{j-1} + \cdots + b_r a_{j-r} \text{ if } j \geq r. \end{cases}$$

According to definition of complexity (see Section 2.1) and to relations (10) and (11), for every $m \in \mathbb{N}$, we have

$$p\left(b(t)f(T), m\right) = p\left(b(T)(\sum_{i\geq 0} a_i T^{-i}), m\right) = p\left((\sum_{j\geq r} c_j T^{-j}), m\right).$$

Our aim is to count the number of words of form $c_j c_{j+1} \cdots c_{j+m-1}$, when $j \geq r$. By definition of $\mathbf{c}$, we notice that for $j \geq r$ these words depend only

on $a_{j-r}a_{j-r+1}\cdots a_{j+m-1}$ and of $b_0, b_1, \cdots, b_r$, which are fixed. The number of words $a_{j-r}a_{j-r+1}\cdots a_{j+m-1}$ is exactly $p(f, m+r)$. By Lemma 2.1 we obtain:

$$p(f, m+r) < p(f, r)p(f, m) = Mp(f, m),$$

where $M = p(f, r)$. More precisely, we may bound up M by q^r, since this is the number of all possible words of length r over an alphabet of q letters. $\square$

Proposition 4.3. *Let $r(T) \in \mathbb{F}_q(T)$ and $f(T) = \sum_{n\geq -n_0} a_n T^{-n} \in \mathbb{F}_q((T^{-1}))$. Then for every $m \in \mathbb{N}$, there is a positive constant M, depending only on r and n_0, such that:*

$$p(rf, m) \leq Mp(f, m).$$

Proof. Let $f(T) := \sum_{i\geq -i_0} a_i T^{-i} \in \mathbb{F}_q((T^{-1}))$, $i_0 \in \mathbb{N}$ and $m \in \mathbb{N}$. By Proposition 4.1, we have:

$$p(r(T)f(T), m) \leq p\left(r(T)(\sum_{i=-i_0}^{-1} a_i T^{-i}), m\right) \cdot p\left(r(T)(\sum_{i\geq 0} a_i T^{-i}), m\right).$$

Proposition 4.2 implies that

$$p\left(r(T)(\sum_{i=-i_0}^{-1} a_i T^{-i}), m\right) \leq R$$

where R does not depend on m. Thus, we may assume that $f(T) = \sum_{i\geq 0} a_i T^{-i}$.

The proof of Proposition 4.3 is decomposed into five steps.

Step 1. Since $r(T) \in \mathbb{F}_q(T)$, the sequence of coefficients of r is eventually periodic. Thus, there exist two positive integers S and L and two polynomials $p_1 \in \mathbb{F}_q[T]$ (with degree equal to $S-1$) et $p_2 \in \mathbb{F}_q[T]$ (with degree equal to $L-1$) such that r may be written as follows:

$$r(T) = \frac{P(T)}{Q(T)} = \frac{p_1(T)}{T^{S-1}} + \frac{p_2(T)}{T^{S+L-1}}(1 + T^{-L} + T^{-2L} + \cdots).$$

Hence

(12)

$$r(T)f(T) = \underbrace{\frac{1}{T^{S-1}} p_1(T)f(T)}_{g(T)} + \underbrace{p_2(T)\frac{1}{T^{S+L-1}} f(T)(1 + T^{-L} + T^{-2L}\cdots)}_{h(T)}$$
$$:= \sum_{n\geq 0} f_n T^{-n}.$$

Let us denote by $\mathbf{d} = (d(n))_{n\geq 0}$ the sequence of coefficients of $g(T)$ and by $\mathbf{e} = (e_n)_{n\geq 0}$ the sequence of coefficients of $h(T)$. Clearly $\mathbf{f} := (f_n)_{n\geq 0}$ is such that $f_n = d_n + e_n$, for every $n \in \mathbb{N}$.

Fix $m \in \mathbb{N}$. Our aim is to bound from above $p(\mathbf{f}, m)$. First, assume that m is a multiple of L. More precisely, we set $m = kL$, where $k \in \mathbb{N}$.

In order to bound the complexity of $\mathbf{f}$, we will consider separately the sequences $\mathbf{e}$ and $\mathbf{d}$.

Step 2. We study now the sequence $\mathbf{e}$, defined in (12).
In order to describe the sequence $\mathbf{e}$, we shall study first the product

$$f(T)(1+T^{-L}+T^{-2L}+\cdots) = (\sum_{i\geq 0} a_i T^{-i})(1+T^{-L}+T^{-2L}+\cdots) := \sum_{j\geq 0} c_j T^{-j}.$$

Expanding this product, it is not difficult to see that: $c_l = a_l$ if $l < L$ and $c_{kL+l} = a_l + a_{l+L} + \cdots + a_{kL+l}$, for $k \geq 1$ and $0 \leq l \leq L-1$.

By definition of c_n, $n \in \mathbb{N}$, we can easily obtain:

$$c_{n+L} - c_n = a_{n+L}.$$

Consequently, for all $s \in \mathbb{N}$:

$$c_{n+sL} - c_n = a_{n+sL} + a_{n+(s-1)L} + \cdots + a_{n+L}. \tag{13}$$

Our goal is now to study the subwords of $\mathbf{c}$ with length $m = kL$.
Let $j \geq 0$ and let $c_j c_{j+1} c_{j+2} \cdots c_{j+kL-1}$ be a finite factor of length $m = kL$. Using identity (13), we may split the factor above in k words of length L as follows:

$$c_j c_{j+1} c_{j+2} \cdots c_{j+kL-1} = \underbrace{c_j c_{j+1} \cdots c_{j+L-1}}_{D_1} \underbrace{c_{j+L} c_{j+L+1} \cdots c_{j+2L-1}}_{D_2} \cdots$$
$$\cdots \underbrace{c_{j+(k-1)L} c_{j+(k-1)L+1} \cdots c_{j+kL-1}}_{D_k}$$

where the words D_i, $2 \leq i \leq k$ depend only on D_1 and $\mathbf{a}$. More precisely, we have:

$$\begin{aligned} D_2 &= (c_j + a_{j+L})(c_{j+1} + a_{j+L+1}) \cdots (c_{j+L-1} + a_{j+2L-1}) \\ &\vdots \\ D_k &= (c_j + a_{j+L} + \cdots + a_{j+(k-1)L})(c_{j+1} + a_{j+L+1} + \cdots + a_{j+(k-1)L+1}) \cdots \\ &\quad (c_{j+L-1} + a_{j+2L-1} + \cdots + a_{j+kL-1}). \end{aligned}$$

Consequently, the word $c_j c_{j+1} c_{j+2} \cdots c_{j+m-1}$ depends only on D_1, which is a factor of length L, determined by $r(T)$, and on the word $a_{j+L} \cdots a_{j+kL-1}$, factor of length $kL - L = m - L$ of $\mathbf{a}$.

Now, let us return to the sequence $\mathbf{e}$. We recall that

$$\sum_{n\geq 0} e_n T^{-n} = \frac{p_2(T)}{T^{S+L-1}} \sum_{j\geq 0} c_j T^{-j}. \tag{14}$$

Using a similar argument as in the proof of Proposition 4.2 and using the identity (14), a factor of the form $e_j e_{j+1} \cdots e_{j+m-1}$, $j \in \mathbb{N}$, depends only on the coefficients of p_2, which are fixed, and on $c_{j-L+1} \cdots c_{j-1} c_j \cdots c_{j+m-1}$. Hence, the number of distinct factors of the form $e_j e_{j+1} \cdots e_{j+m-1}$ depends only on the number of distinct factors of the form $a_{j+1} a_{j+2} \cdots a_{j+(k-1)L}$ and on the number of factors of length L that occur in $\mathbf{c}$.

Step 3. We describe now the sequence $\mathbf{d}$, defined in (12).

Doing the same proof as for Proposition 4.2, we obtain that the number of words $d_j \cdots d_{j+m-1}$, when $j \in \mathbb{N}$, depends only on the coefficients of p_1, which are fixed, and on the number of distinct factors $a_{j-S+1} \cdots a_j \cdots a_{j+m-1}$.

Step 4. We now give an upper bound for the complexity of $\mathbf{f}$, when m is a multiple of L.

According to steps 2 and 3, the number of distinct factors of the form $f_j f_{j+1} \cdots f_{j+m-1}$, $j \in \mathbb{N}$, depends on the number of distinct factors of form $a_{j-S+1} a_{j+2} \cdots a_{j+m-1}$ and on the number of factors of length L that occur in $\mathbf{c}$.

Consequently,

$$p(rf, m) \leq p(f, m+S-1) q^L,$$

and by Lemma 2.1

$$p(f, m+S-1) \leq p(f, m) p(f, S-1) \leq q^{S-1} p(f, m).$$

Finally,

$$p(rf, m) \leq q^{L+S-1} p(f, m).$$

Step 5. We now give an upper bound for the complexity of $\mathbf{f}$, when m is not a multiple of L.

In this case, let us suppose that $m = kL + l$, $1 \leq l \leq L-1$. Using Lemma 2.1 and according to Step 4:

$$\begin{aligned} p(rf, m) &= p(rf, kL + l) \leq p(rf, kL) p(rf, l) \leq p(rf, kL) p(rf, L-1) \\ &\leq q^{L-1} p(rf, kL) \leq q^{S+2L-2} p(f, m). \end{aligned}$$

□

4.1.1. *A criterion for linear independence of Laurent series.* As a consequence of Theorem 1.4, we give a criterion of linear independence over $\mathbb{F}_q(T)$ for two Laurent series in function of their complexity.

Proposition 4.4. *Let $f, g \in \mathbb{F}_q((T^{-1}))$ be two irrational Laurent series such that:*

$$\lim_{m \to \infty} \frac{p(f, m)}{p(g, m)} = \infty.$$

Then f and g are linearly independent over the field $\mathbb{F}_q(T)$.

Proof. We argue by contradiction. Assume there exist polynomials $A(T)$, $B(T)$, $C(T)$ over $\mathbb{F}_q$, not all zeros, such that:

$$A(T)f(T) + B(T)g(T) + C(T) = 0.$$

Next use the fact that addition with a rational function and multiplication by a rational function do not increase the asymptotic order of complexity. Indeed, since $A(T) \neq 0$ because $g(T) \notin \mathbb{F}_q(T)$, we would have

$$f(T) + \frac{C(T)}{A(T)} = -\frac{B(T)}{A(T)} g(T).$$

However, Propositions 4.1 and 4.3 would imply that the complexity of the left-hand side of this inequality is asymptotically larger than the one of the right-hand side. $\square$

Let us now give an example of two Laurent series linearly independent over $\mathbb{F}_q(T)$. Their sequences of coefficients are generated by non-uniform morphisms and we study their subword complexity in function of the order of growth of letters, using a classical result of Pansiot [30]. Notice that, the following sequences are non-automatic and hence, the associated Laurent series are transcendental over $\mathbb{F}_q(T)$.

Example 4.1. Consider the infinite word $\mathbf{a} = 000100010001110\cdots$; $\mathbf{a} = (a_n)_{n\geq 0} = \sigma^\infty(0)$ where $\sigma(0) = 0001$ and $\sigma(1) = 11$. If we look to the order of growth of 0 and 1 we have that $|\sigma^n(0)| = 3^n + 5 \cdot 2^{n-2}$ and $|\sigma^n(1)| = 2^n$. Hence, the morphism σ is exponentially diverging (see the Section (2.3)). Consequently, by Pansiot's theorem mentioned above, $p(\mathbf{a}, m) = \Theta(m \log m)$.

Consider next $\mathbf{b} = 010110101111010\cdots$; $\mathbf{b} = (b_n)_{n\geq 0} = \phi^\infty(0)$, where $\phi(0) = 0101$ and $\phi(1) = 11$. It is not difficult to see that ϕ is polynomially diverging (see Section (2.3)) since $|\phi^n(0)| = (n+1)2^n$ and $|\phi(1)^n| = 2^n$. By Pansiot's theorem, $p(\mathbf{b}, m) = \Theta(m \log\log m)$.

Now we consider the formal series whose coefficients are the sequences generated by the morphisms above:

$$f(T) = \sum_{n\geq 0} a_n T^{-n} = \frac{1}{T^3} + \frac{1}{T^7} + \frac{1}{T^{11}} + \frac{1}{T^{12}} + \cdots \in \mathbb{F}_q[[T^{-1}]]$$

and

$$g(T) = \sum_{n\geq 0} b_n T^{-n} = \frac{1}{T^1} + \frac{1}{T^3} + \frac{1}{T^4} + \frac{1}{T^6} + \cdots \in \mathbb{F}_q[[T^{-1}]].$$

Since $\lim_{m\to\infty} p(f, m)/p(g, m) = +\infty$, Proposition 4.4 implies that f and g are linearly independent over $\mathbb{F}_q(T)$.

4.2. **Other closure properties.** In this section we prove that the classes $\mathcal{P}$ and $\mathcal{Z}$ are closed under a number of actions such as: Hadamard product, formal derivative and Cartier operator.

4.2.1. *Hadamard product.* Let $f(T) := \sum_{n\geq -n_1} a_n T^{-n}$, $g(T) := \sum_{n\geq -n_2} b_n T^{-n}$ be two Laurent series in $\mathbb{F}_q((T^{-1}))$. The Hadamard product of f and g is defined as follows:

$$f \odot g = \sum_{n\geq -\min(n_1,n_2)} a_n b_n T^{-n}.$$

As in the case of addition of two Laurent series (see Proposition 4.1) one can easily obtain the following.

Proposition 4.5. *Let f and g be two Laurent series belonging to $\mathbb{F}_q((T^{-1}))$. Then, for every $m \in \mathbb{N}$, we have:*

$$\frac{p(f,m)}{p(g,m)} \leq p(f \odot g, m) \leq p(f,m)p(g,m).$$

The proof is similar to the one of Proposition 4.1. The details are left to the reader.

4.2.2. *Formal derivative.* As an easy application of Proposition 4.5, we present here the following result. First, let us recall the definition of the formal derivative.

Definition 4.1. *Let $n_0 \in \mathbb{N}$ and consider the Laurent series: $f(T) = \sum_{n=-n_0}^{+\infty} a_n T^{-n} \in \mathbb{F}_q((T^{-1}))$. The formal derivative of f is defined as follows:*

$$f'(T) = \sum_{n=-n_0}^{+\infty} (-n \ mod \ p) a_n T^{-n+1} \in \mathbb{F}_q((T^{-1})).$$

We prove then the following result.

Proposition 4.6. *Let $f(T) \in \mathbb{F}_q((T^{-1}))$ and k be a positive integer. If $f^{(k)}$ is the derivative of order k of f, then there exists a positive constant M, such that, for all $m \in \mathbb{N}$, we have:*

$$p(f^{(k)}, m) \leq M\ p(f,m).$$

Proof. The derivative of order k of f is almost the Hadamard product of the series by a rational function. By definition of $p(f,m)$, we may suppose that $f(T) := \sum_{n\geq 0} a_n T^{-n} \in \mathbb{F}_q[[T^{-1}]]$. Then:

$$f^{(k)}(T) = \sum_{n\geq k} ((-n)(-n-1)\cdots(-n-k+1)a_n)T^{-n-k} := T^{-k} \sum_{n\geq k} b_n a_n T^{-n},$$

where $b_n := (-n)(-n-1)\cdots(-n-k+1) \mod p$. Since $b_{n+p} = b_n$, the sequence $(b_n)_{n\geq 0}$ is periodic of period p. Hence, let us denote by $g(T)$ the series whose coefficients are precisely given by $(b_n)_{n\geq 0}$. Thus there exists a positive constant M such that:

$$p(g,m) \leq M.$$

By Proposition 4.5,

$$p(f^{(k)}, m) \leq p(g,m)p(f,m) \leq Mp(f,m),$$

which completes the proof. □

4.2.3. *Cartier's operators.* In the fields of positive characteristic, there is a natural operator, the so-called "Cartier operator" that plays an important role in many problems in algebraic geometry and arithmetic in positive characteristic [16, 17, 23, 32]. In particular, if we consider the field of Laurent series with coefficients in $\mathbb{F}_q$, we have the following definition.

Definition 4.2. *Let $f(T) = \sum_{i\geq 0} a_i T^{-i} \in \mathbb{F}_q[[T^{-1}]]$ and $0 \leq r < q$. The Cartier operator Λ_r is a linear transformation such that:*

$$\Lambda_r(\sum_{i\geq 0} a_i T^{-i}) = \sum_{i\geq 0} a_{qi+r} T^{-i}.$$

The classes $\mathcal{P}$ and $\mathcal{Z}$ are closed under this operator. More precisely, we prove the following result.

Proposition 4.7. *Let $f(T) \in \mathbb{F}_q[[T^{-1}]]$ and $0 \leq r < q$. Then there is M such that, for every $m \in \mathbb{N}$ we have the following:*

$$p(\Lambda_r(f), m) \leq qp(f, m)^q.$$

Proof. Let $\mathbf{a} := (a_n)_{n\geq 0}$ be the sequence of coefficients of f and $m \in \mathbb{N}$. In order to compute $p(\Lambda_r(f), m)$, we have to look at factors of the form

$$a_{qj+r} a_{qj+q+r} \cdots a_{qj+(m-1)q+r},$$

for all $j \in \mathbb{N}$. But these only depend on factors of the form

$$a_{qj+r} a_{qj+r+1} \cdots a_{qj+(m-1)q+r}.$$

Using Lemma 2.1, we obtain that:

$$p(\Lambda_r(f), m) \leq p(f, (m-1)q + 1) \leq qp(f, m-1)^q \leq qp(f, m)^q.$$

□

5. Cauchy product of Laurent series

In the previous section, we proved that $\mathcal{P}$ and $\mathcal{Z}$ are vector space over $\mathbb{F}_q(T)$. This raises naturally the question whether or not these classes form a ring; i.e., are they closed under the usual Cauchy product? There are actually some particular cases of Laurent series with low complexity whose product still belongs to $\mathcal{P}$. In this section we discuss the case of automatic Laurent series. However, we are not able to prove whether $\mathcal{P}$ or $\mathcal{Z}$ are or not rings or fields.

5.1. **Products of automatic Laurent series.** A particular case of Laurent series stable by multiplication is the class of k-automatic series, k being a positive integer:

$$\mathrm{Aut}_k = \{f(T) = \sum_{n\geq 0} a_n T^{-n} \in \mathbb{F}_q((T^{-n})),\, \mathbf{a} = (a_n)_{n\geq 0} \text{ is } k\text{-automatic}\}.$$

Since any k-automatic sequence has at most a linear complexity, $\mathrm{Aut}_k \subset \mathcal{P}$. A theorem of Allouche and Shallit [9] states that the set Aut_k is a ring.

In particular, this implies that, if f and g belong to Aut_k, then $p(fg,m) = O(m)$. Notice also that, in the case where k is a power of p, the characteristic of the field $\mathbb{F}_q((T^{-1}))$, the result follows from Christol's theorem.

Remark 5.1. However, we do not know whether or not this property is still true if we replace Aut_k by $\cup_{k\geq 2}\mathrm{Aut}_k$. More precisely, if we consider two Laurent series $f, g \in (\cup_{k\geq 2}\mathrm{Aut}_k)$ we do not know if the product fg is still in $\mathcal{P}$. The next subsection gives a particular example of two Laurent series belonging to $(\cup_{k\geq 2}\mathrm{Aut}_k)$ and such that the product fg is still in $\mathcal{P}$.

5.1.1. *Some lacunary automatic Laurent series.* We consider now some particular examples of lacunary series. More precisely, we focus on the product of series of form:

$$f(T) = \sum_{n\geq 0} T^{-d^n} \in \mathbb{F}_q((T^{-1})).$$

It is not difficult to prove that $p(f,m) = O(m)$. The reader may refer to [25] for more general results concerning the complexity of lacunary series. The fact that the complexity of f is linear is implied also by the fact that $f \in \mathrm{Aut}_d$. Notice also that f is transcendental over $\mathbb{F}_q(T)$ if q is not a power of d. This is an easy consequence of Christol's theorem and a theorem of Cobham [19].

In this section we will prove the following result.

Theorem 5.1. *Let d and e be two multiplicatively independent positive integers (that is $\frac{\log d}{\log e}$ is irrational) and let $f(T) = \sum_{n\geq 0} T^{-d^n}$ and $g(T) = \sum_{n\geq 0} T^{-e^n}$ be two Laurent series in $\mathbb{F}_q((T^{-1}))$. Then:*

$$p(fg,m) = O(m^4).$$

Remark 5.2. The series f and g are linearly independent over $\mathbb{F}_q(T)$. More generally, any two irrational Laurent series, d-automatic and respectively e-automatic, where d and e are two multiplicatively independent positive integers, are linearly independent over $\mathbb{F}_q(T)$. This follows by a Cobham's theorem.

Let us denote by $h(T) := f(T)g(T)$. Then $h(T) = \sum_{n\geq 0} a_n T^{-n}$ where the sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is defined as follows:

$$a_n := (\text{the number of pairs } (k,l) \in \mathbb{N}^2 \text{ that verify } n = d^k + e^l) \mod p.$$

The main clue of the proof is the following consequence of the theory of S-unit equations (see [2] for a proof).

Lemma 5.1. *Let d and e be two multiplicatively independent positive integers. There is a finite number of solutions $(k_1, k_2, l_1, l_2) \in \mathbb{N}^4$, $k_1 \neq k_2$, $l_1 \neq l_2$, that satisfy the equation:*

$$d^{k_1} + e^{l_1} = d^{k_2} + e^{l_2}.$$

Obviously, we have the following consequence concerning the sequence $\mathbf{a} = (a_n)_{n\geq 0}$:

Corollary 5.1. *There exists a positive integer N such that, for every $n \geq N$ we have $a_n \in \{0, 1\}$. Moreover, $a_n = 1$ if and only if there exists one unique pair $(k, l) \in \mathbb{N}^2$ such that $n = d^k + e^l$.*

We prove now the Theorem 5.1. For the sake of simplicity, we consider $d = 2$ and $e = 3$, but the proof is exactly the same in the general case.

Proof. Let $\mathbf{b} := (b_n)_{n\geq 2}$ and $\mathbf{c} := (c_n)_{n\geq 2}$ be the sequences defined as follows:

$$b_n = \begin{cases} 1 & \text{if there exists a pair } (k,l) \in \mathbb{N}^2 \text{ such that } n = 2^k + 3^l,\ 2^k > 3^l; \\ 0 & \text{otherwise,} \end{cases}$$

$$c_n = \begin{cases} 1 & \text{if there exists a pair } (k,l) \in \mathbb{N}^2 \text{ such that } n = 2^k + 3^l,\ 2^k < 3^l; \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by $h_1(T) := \sum_{n\geq 2} b_n T^{-n}$ and resp. $h_2(T) := \sum_{n\geq 2} c_n T^{-n}$ the series associated to $\mathbf{b}$ and $\mathbf{c}$. Using Corollary 5.1, there exists a polynomial $P \in \mathbb{F}_q[T]$, with degree less than N, such that h can be written as follows:

$$h(T) = h_1(T) + h_2(T) + P(T).$$

By Remark 4.2, there is $C \in \mathbb{R}$ such that, for any $m \in \mathbb{N}$:

$$p(h, m) \leq p(h_1 + h_2, m) + C.$$

In the sequel, we will show that $p(h_1, m) = p(h_2, m) = O(m^2)$. Theorem 5.1 will then follow by Proposition 4.1.

We study now the subword complexity of the sequence of coefficients $\mathbf{b} := (b_n)_{n\geq 2}$. The proof is similar to the proofs of Theorems 1.2 and 1.3. The complexity of the sequence $\mathbf{c}$ can be treated in essentially the same way as for $\mathbf{b}$.

Step 1. For all $n \geq 1$, we denote by W_n the factor of $\mathbf{b}$ that occurs between positions $2^n + 1$ and 2^{n+1}, that is:

$$W_n := b_{2^n+3^0} b_{2^n+2} b_{2^n+3^1} \cdots b_{2^{n+1}}.$$

We also set $W_0 := 1$.

Observe that $|W_n| = 2^n$.

With these notations the infinite word $\mathbf{b}$ can be factorized as:

$$\mathbf{b} = \underbrace{1}_{W_0}\underbrace{10}_{W_1}\underbrace{1010}_{W_2}\underbrace{10100000}_{W_3}\cdots = W_0W_1W_2\cdots . \tag{15}$$

Step 2. Let $n \geq 1$ and m_n be the greatest integer such that $2^n + 3^{m_n} \leq 2^{n+1}$. This is equivalently to say that m_n is such that

$$2^n + 3^{m_n} < 2^{n+1} < 2^n + 3^{m_n+1}.$$

Notice also that $m_n = n\lfloor \log_3 2\rfloor$.

With these notations we have (for $n \geq 5$):

$$W_n = 1010^5 1 \cdots 10^{\alpha_i} \cdots 10^{\alpha_{m_n}} 10^{\beta_n},$$

where $\alpha_i = 2 \cdot 3^{i-1} - 1$, for $1 \leq i \leq m_n$, and $\beta_n = 2^n - 3^{m_n} \geq 0$.

Let us denote by U_n the prefix of W_n such that $W_n := U_n 0^{\beta_n}$.

Notice that $(m_n)_{n\geq 0}$ is an increasing sequence. Hence $(\alpha_{m_n})_{n\geq 0}$ is increasing. Consequently, $U_n \prec_p U_{n+1}$ and more generally, $U_n \prec_p W_i$, for every $i \geq n+1$.

Step 3. Let $M \in \mathbb{N}$. Our aim is to bound from above the number of distinct factors of length M occurring in $\mathbf{b}$. In order to do this, we will show that there exists an integer N such that all these factors occur in

$$W_0W_1\cdots W_N$$

or in the set

$$A_0 := \{Z \in \mathcal{A}^M;\ Z \text{ is of the form } 0^j P \text{ or } 0^i 10^j P,\ P \prec_p U_N,\ i,j \geq 0,\}.$$

Let $N = \lceil \log_2(M+1)\rceil + 3$. Doing a simple computation we obtain that $\alpha_{m_N} \geq M$. Notice also that, for any $i \geq N$ we have

$$\alpha_{m_i} \geq M.$$

This follows since $(\alpha_{m_n})_{n\geq 0}$ is an increasing sequence.

Let V be a factor of length M of $\mathbf{b}$. Suppose that V does not occur in the prefix $W_0W_1\cdots W_N$. Then, by (15), V must occur in $W_NW_{N+1}\cdots$. Hence, V must appear in some W_i, for $i \geq N+1$, or in $\bigcup_{i\geq N} i(W_i, \varepsilon, W_{i+1})$.

Let us suppose that V occurs in $\bigcup_{i\geq N} i(W_i, \varepsilon, W_{i+1})$. Since W_i ends with $0^{\alpha_{m_i}}10^{\beta_i}$, with $\alpha_{m_i} \geq M$, and since W_{i+1} begins with U_N and $|U_N| = 3^{m_N} + 1 \geq M$, we have that

$$\mathcal{A}^M \cap (\bigcup_{i\geq N} i(W_i, \varepsilon, W_{i+1})) \subset A_0.$$

Hence, if V occurs in $\bigcup_{i\geq N} i(W_i, \varepsilon, W_{i+1})$ then $V \in A_0$.

Let us suppose now that V occurs in some W_i, for $i \geq N+1$. By definition of W_i and α_i, for $i \geq N+1$ and by the fact that we have:

$$W_i = 1010^5 1 \cdots 10^{\alpha_{m_N}} 10^{\alpha_{m_N+1}} \cdots 10^{\alpha_{m_i}} 10^{\beta_i} = U_N 0^{\alpha_{m_N+1}} \cdots 10^{\alpha_{m_i}} 10^{\beta_i}.$$

By assumption, V does not occur in $W_0 W_1 \cdots W_N$; hence V cannot occur in U_N which by definition is a prefix of W_N. Consequently, V must be of the form $0^r 10^s$, $r, s \geq 0$. Indeed, since $\alpha_{m_N} \geq M$, all blocks of zeros that follow after U_N (and before the last digit 1 in W_i) are all longer than M. But the words of form $0^r 10^s$, $r, s \geq 0$ belong also to A_0.

Hence, we proved that if V does not occur in the prefix $W_0 W_1 \cdots W_N$, then V belongs to A_0, as desired.

Step 4. In the previous step we showed that all distinct factors of length M occur in the prefix $W_0 W_1 \cdots W_N$ or in the set A_0.

Since

$$|W_0 W_1 \cdots W_N| = \sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$

and since $N = \lceil \log_2(M+1) \rceil + 3$ we have that:

$$2^{N+1} - 1 \leq 2^{\log_2(M+1)+5} - 1 = 32M + 31,$$

and the number of distinct factors that occur in $W_0 W_1 \cdots W_N$ is less or equal to $32M + 31$.

Also, by an easy computation, we obtain that the cardinality of the set A_0 is

$$\operatorname{Card} A_0 = \frac{M^2}{2} + \frac{3M}{2}.$$

Finally, $p(\mathbf{b}, m) = p(h_1, m) = O(m^2)$. In the same manner, one could prove that $p(h_2, m) = O(m^2)$. This achieves the proof of Theorem 5.1, in view of Proposition 4.1. □

5.2. **A more difficult case.** Set

$$\theta(T) := 1 + 2 \sum_{n \geq 1} T^{-n^2} \in \mathbb{F}_q((T^{-1})),\ q \geq 3.$$

The function $\theta(T)$ is related to the classical Jacobi theta function. The sequence of coefficients of $\theta(T)$ corresponds to the characteristic sequence of squares and one can easily prove that:

$$p(\theta, m) = \Theta(m^2).$$

In particular this implies the transcendence of $\theta(T)$ over $\mathbb{F}_q(T)$, for any $q \geq 3$. Notice that this implies the transcendence over $\mathbb{Q}(T)$ of the same Laurent series but viewed as an element of $\mathbb{Q}((T^{-1}))$.

Since $\theta(T) \in \mathcal{P}$, it would be interesting to know whether or not $\theta^2(T)$ belongs also to $\mathcal{P}$. Notice that

$$\theta^2(T) = \sum_{n \geq 1} r_2(n) T^{-n}$$

where $r_2(n)$ is the number of representations of n as sum of two squares of integers mod p.

In the rich bibliography concerning Jacobi theta function (see for instance [24, 27]), there is the following well-known formula:

$$r_2(n) = 4(d_1(n) - d_3(n)) \bmod p$$

where $d_i(n)$ denotes the number of divisors of n congruent to i modulo 4, for each $i \in \{1, 3\}$.

In particular, by an easy consequence of Fermat's 2-squares theorem we can deduce that $r_2(n) = 0$ if n is a prime of the form $4k + 3$ and $r_2(n) = 8$ mod p if n is a prime of the form $4k + 1$.

More generally, if

$$n = 2^{\gamma} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} q_1^{\beta_1} q_2^{\beta_2} \cdots q_l^{\beta_l},$$

where $p_i \equiv 1$ [4] et $q_j \equiv 3$ [4] then

$$r_2(n) = \begin{cases} 0 \text{ if there exists an odd } \beta_j \text{ in the decomposition of } n; \\ 4(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) \quad \bmod p \text{ if all } \beta_j \text{ are even.} \end{cases}$$

Using these properties, we may easily deduce that $r_2(n)$ is a multiplicative function of n. Recall that we would like to study the subword complexity of $r_2(n)_{n \geq 0}$, that is the number of distinct factors of form $r_2(j)r_2(j+1) \cdots r_2(j+m-1)$, when $j \in \mathbb{N}$. Hence, it would be useful to describe some additive properties of $r_2(n)_{n \geq 0}$; for instance, it would be interesting to find some relations between $r_2(j + N)$ and $r_2(j)$, for some positive integers j, N. This seems to be a rather difficult question about which we are not able to say anything conclusive.

6. Conclusion

It would be also interesting to investigate the following general question.

Is it true that Carlitz's analogs of classical constants all have a "low" complexity (i.e., polynomial or subexponential)?

The first clue in this direction are the examples provided by Theorems 1.1, 1.2 and 1.3. Notice also that a positive answer would reinforce the differences between $\mathbb{R}$ and $\mathbb{F}_q((T^{-1}))$ as hinted in our introduction. When investigating these problems, we need, in general, the Laurent series expansions of such functions. In this context, one has to mention the work of Berthï¿½ [12, 13, 14, 15], where some Laurent series expansions of Carlitz's functions are described.

When a Laurent series has a "low" complexity, the combinatorial structure of its sequence of coefficients is rich and this can be used to derive some interesting Diophantine properties. Using this principle, bounds for irrationality

measures can be obtained for elements of the class of Laurent series with at most linear complexity.

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